Math 565: Functional Analysis
Lecture 25

For a normal operator $T \in B(H)$ with $T = T_1 + iT_2$, T_i self-adjoint, it suffices to simultaniously diagonalize T_1 and T_2, so it's enough to prove the following:

Thm. Let $\mathcal{A} \subseteq B_c(H)$ be a set of commuting compact self-adjoint operators. Then $\exists$ ON basis for H consisting of eigenvectors of all of $\mathcal{A}$.

Proof. First let's prove this for $\mathcal{A} = \{T, S\}$. By the spectral theorem for T, $H = \bigoplus_{\lambda \in \sigma_p(T)} E_\lambda$ and each E_λ is finite dimensional. The key observation is that E_λ is S-invariant: for $x \in E_\lambda$, we have $T(Sx) = S(Tx) = S(\lambda x) = \lambda S(x)$ so $S(x) \in E_\lambda$. Applying the spectral theorem to $S|_{E_\lambda}$, we get an ON basis B_λ for E_λ consisting of eigenvectors of S. Thus, $\bigcup_{\lambda \in \sigma_p(T)} B_\lambda$ is an ON basis of common eigenvectors for S and T. For general $\mathcal{A}$, take a $T \in \mathcal{A}$ w/ $H = \bigoplus_{\lambda \in \sigma_p(T)} E_\lambda$ as above, the split each E_λ maximally into a direct sum of spaces that are common subspaces of eigenspaces for all of $\mathcal{A}$, details left as an exercise. $\square$

Cor (Spectral thm for compact normal operators). Every compact normal $T \in B(H)$ admits an ON basis of eigenvectors.

Proof. $T = T_1 + iT_2$, where T_1, T_2 are commuting self-adjoint compact operators, so we apply the prev. thm to $\mathcal{A} := \{T_1, T_2\}$. $\square$

Def. Let H_1, H_2 be Hilbert spaces and $T_i \in B(H_i)$ for $i = 1, 2$. We say that T_1 and T_2 are *unitarily equivalent* if $\exists$ unitary $U : H_1 \to H_2$ such that $T_2 = U T_1 U^{-1}$.

Cor. Every compact normal $T \in B(H)$ is unitarily equivalent to a multiplication operator M_λ on $\ell^2(\text{Hdim}(H))$, where $\lambda \in \ell^\infty(\text{Hdim}(H))$ and $\lim_{i \to \infty} \lambda(i) = 0$.

$\Longleftrightarrow$

General spectral theory

Spectrum of an operator.

General, even self-adjoint, bdd operators on Hilbert spaces may not have any eigenvalues.

Example. Let (X,μ) be a semifinite measure space, i.e. every positive measure set contains a positive finite measure subset. For any $\varphi \in L^\infty(X,\mu)$, the multiplication op. M_φ on $L^2(X,\mu)$ admits a $\lambda \in \mathbb{C}$ as an eigenvalue $\Leftrightarrow$ $\varphi^{-1}(\lambda)$ has positive measure. Indeed, $\Rightarrow$ follows from the fact that if $\varphi f = \lambda f$ then $\{f \neq 0\} \subseteq \varphi^{-1}(\lambda)$ a.e., while $\Leftarrow$ follows by semifiniteness of μ since we can take $B \subseteq \varphi^{-1}(\lambda)$ of positive finite measure so $\mathbb{1}_B \in L^2(X,\mu)$ and $\varphi \cdot \mathbb{1}_B = \lambda \mathbb{1}_B$.

In particular, if $(X,\mu) = (K, \text{Lebesgue})$ where $K \subseteq \mathbb{C}$ compact nonempty set, then the mult. op. M_φ, where $\varphi := id_K$, i.e. $\varphi(x) = x$, has no eigenvalues.

Definition. Let X be a Banach space and let $T \in B(X)$.

- The **resolvent set** of T is $\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is invertible in } B(X)\}$.
- The **spectrum** of T is $\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in } B(X)\}$. The **spectral radius** of T is $\|T\|_\sigma := \sup\{|\lambda| : \lambda \in \sigma(T)\}$.

Examples. (a) If $\dim(X) < \infty$, then $\sigma(T) = \sigma_p(T)$ by the rank-nullity theorem.

(b) Let H be a Hilbert space and $\{e_i\}_{i \in I}$ be an ON basis. Let $T \in B(H)$ be defined by $Te_i = \lambda_i e_i$ s.t. $\lim_{i \to \infty} \lambda_i = 0$, so T is compact. Then $\sigma_p(T) = \{\lambda_i : i \in I\}$. However, even if all $\lambda_i \neq 0$, we still have $0 \in \sigma(T)$, in fact $\sigma(T) = \{0\} \cup \sigma_p(T)$. Indeed, T is not invertible (but surjective) because otherwise, $T^{-1} e_i = \lambda_i^{-1} e_i$, and $\lim_{i \to \infty} \lambda_i^{-1} = \infty$ so $\|T^{-1}\| = \infty$.

(c) For any multiplication op. M_φ on $L^2(X,\mu)$ for some $\varphi \in L^\infty(X,\mu)$ and (X,μ) any measure space, $\sigma(M_\varphi) = \operatorname{essran}(\varphi)$.

Thm. For any Banach space X and $T \in B(X)$, $\sigma(T)$ is a compact nonempty subset of $\mathbb{C}$. In fact, $\|T\|_\sigma \le \|T\|$.

Proof. The inequality $\|T\|_\sigma \le \|T\|$ and that $\rho(T)$ is open, hence $\sigma(T)$ is compact, is in HW6, so we only argue for $\sigma(T) \neq \emptyset$. For this we will need complex analysis of Banach space valued functions on $\mathbb{C}$.

Def. Let $\Omega \subseteq \mathbb{C}$ be open and X be a (complex) Banach space. A function $f : \Omega \to X$ is called *analytic* if for each $z_0 \in \Omega$ $\exists\, r > 0$ such that $\forall\, z \in B_r^{\mathbb{C}}(z_0)$,

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n x_n,$$

for some $\{x_n\}_{n \in \mathbb{N}} \subseteq X$, where this series converges absolutely.
This is equivalent to saying that $\forall\, z_0 \in \Omega$, the derivative $f'(z_0) := \lim_{h \to 0} \frac{1}{h}(f(z_0+h) - f(z_0))$ exists. (This equivalence requires development of the theory.)

Everything in complex analysis holds, including Liouville's theorem: if $f : \mathbb{C} \to X$ is analytic and bdd then it constant.

Now in HW6, we show the function $f : \rho(T) \to B(X) : \lambda \mapsto (T - \lambda I)^{-1}$ is analytic (by the Taylor series definition) and if $\rho(T) = \mathbb{C}$, then f is also bdd, hence constant by Liouville's thm. But f is not constant, so $\rho(T) \neq \mathbb{C}$, hence $\sigma(T) \neq \emptyset$. $\square$

Thm (useful in many subjects). For any $T \in B(X)$, $\|T\|_\sigma = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}}$.

Proof. Uses radius of convergence of Taylor series, will be outlined as a problem. $\square$

Cor. If T is a bdd normal op. on a Hilbert space H, then $\|T^n\| = \|T\|^n$, hence $\|T\|_\sigma = \|T\|$.

Proof. To see $\|T^2\| = \|T\|$, use $\langle T^2 x, T^2 x \rangle = \langle T^*Tx, T^*Tx \rangle$ and use that T^*T is self-adjoint so $\|T^*T\| = \sup_{\|x\|=1} |\langle T^*Tx, x \rangle|$. $\square$

Spectral theory for self-adjoint and normal operators.

Spectral theorem for self-adjoint operators. Let H be an arbitrary Hilbert space. Every self-adjoint op. $T \in B(H)$ is unitarily equiv. to a multip. op. M_φ on $L^2(X,\mu)$, where $\varphi \in L^\infty(X,\mu)$ and (X,μ) is a semifinite measure space

We first prove this in a special case where T has a cyclic vector.

Def. For $T \in B(H)$, a vector $x \in H$ is called *T-cyclic* if H is the smallest T-invariant closed subspace of H; equivalently, $\{p(T)x : p \in \mathbb{C}[t] \text{ polynomial}\}$ is dense in H.

In a finite dimensional setting, having a cyclic vector implies that $\dim(E_\lambda) = 1$ $\forall \lambda \in \sigma(T)$.
In ∞-dim, we still have "no geometric multiplicity" for $\sigma(T)$.

Spectral thm for cyclic self-adjoint operators. Every self-adjoint $T \in B(H)$ which admits a cyclic vector is unitarily equiv. to the multiplication op. M_φ on $L^2(\sigma(T),\mu)$, where $\varphi := id$, i.e. $\varphi(\lambda) = \lambda$, and μ is a Radon probability measure on $\sigma(T)$.

Proof idea. (1) Let $\mathcal{A}_T := \overline{\{p(T) : p \in \mathbb{C}[t]\}}$, so it is a commutative C^*-algebra. Put $X := \sigma(T)$. Then there is a unique C^*-algebra isomorphism $f \mapsto T_f : C(X) \xrightarrow{\sim} \mathcal{A}_T$ mapping the identity id_X to T; indeed, $id_X \mapsto T$ extends to $\mathbb{C}[id_X] \to \mathbb{C}[T]$, which extends to $C(X) \to \mathcal{A}_T$ by the Stone–Weierstrass theorem.
(2) Let $v \in H$ be a T-cyclic vector with $\|v\| = 1$ and check that $\ell : C(X) \to \mathbb{C} : f \mapsto \langle T_f v, v\rangle$ is a bdd linear positive functional on $C(X)$ with $\|\ell\| = 1$. Since X is compact Hausdorff, Riesz representation thm says that $\ell = I_\mu$ for some Radon prob. meas. μ on X. Thus, $\langle T_f v, v\rangle = \int_X f\,d\mu$ for each $f \in C(X)$.
(3) There is a (unique) unitary $U : L^2(X,\mu) \to H$ such that $Uf = T_f v$ for all $f \in C(H)$.
(4) It remains to verify that $U^{-1} T_f U = M_f$ for all $f \in C(X)$. $\square$

Cyclic decomposition. For each self-adjoint $T \in B(H)$, there is a decomposition $H = \bigoplus_{i\in I} H_i$ into T-invariant closed subspaces $H_i \subseteq H$ s.t. $T|_{H_i}$ has a cyclic vector.

Proof. Zorn's lemma gives a maximal closed T-invariant subspace $M \subseteq H$ which admits a decomposition $M = \bigoplus_{i \in I} H_i$ as in the statement. Maximality then implies that $M^\perp = 0$, so $M = H$. □

Putting the last two theorems together, one obtains the general spectral theorem for all self-adjoint operators on a Hilbert space. As with compact operators, this can be boosted to all normal operators (hence also for all unitary operators).

Spectral theorem for normal operators. Let H be an arbitrary Hilbert space. Every normal operator $T \in B(H)$ is unitarily equivalent to a multip. op. M_φ on $L^2(X,\mu)$, where $\varphi \in L^\infty(X,\mu)$ and (X,μ) is a semifinite measure space

Proof idea. The proof for self-adjoint operators would go through if we could show that the commutative C^*-algebra $\mathcal{A}_T$ generated by T, i.e. the closure of $\{p(T,T^*) : p \in \mathbb{C}[T,T^*]\}$, is isomorphic to $C(X)$ for some compact X. Indeed, one could then get the same cyclic decomposition for $\mathcal{A}_T$ (instead of just T) and run the proof of the case of the cyclic case, as before. Such a $C(X)$ is obtained via Gelfand transform: X is a certain closed subset of the closed unit ball $\{\ell \in \mathcal{A}_T^* : \|\ell\| \leq 1\}$ of the dual of $\mathcal{A}_T$ with the weak* topology, so it's compact and the restriction map $S \mapsto \hat{S}|_X : \mathcal{A}_T \to C(X)$ is an isomorphism of C^*-algebras. □